Earnings growth and the wealth distribution

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As measured by Gini coefficients, fractile inequalities, and tail power laws, wealth is distributed less equally across people than are labor earnings. We study how luck, attitudes that shape saving decisions, and growth rates of labor earnings balance each other in ways that simultaneously shape joint distributions across people of labor earnings, age, and wealth together with an equilibrium rate of return on savings that plays a pivotal role in balancing contending forces. Strong motives for people to save and for firms to demand capital raise an equilibrium interest rate enough to make wealth grow faster than labor earnings. That makes cross-sectional wealth more unevenly distributed and have a fatter tail than labor earnings, as in US data.

inequality | power law | heavy tail

Forces and Main Results

We begin with a streamlined setting, in which an equilibrium interest rate and distribution of wealth depend on preferences and opportunities in a continuous-time economy populated by a unit measure of ex ante identical, but ex post heterogeneous, agents who have random life spans. After isolating forces that generate wealth inequality, we investigate how adding sources of ex ante heterogeneity alters equilibrium wealth distributions.* Our model makes cross-sectional wealth more unequal and fatter-tailed than labor earnings, as is true in US data.

Choices. A person is born at age 0 and dies at a random nonnegative age τ that is exponentially distributed with a constant death rate λ per unit of time, as in ref. 5. An agent ranks consumption processes $\{C_t\}_{t=0}^{\infty}$ by

$$\mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\, U(C_t)\, dt\right], \qquad [1]$$

where $\rho > 0$ is a discount rate, $\mathbb{E}[\,\cdot\,]$ is a mathematical expectation with respect to the probability distribution of τ, and

$$U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \text{if } \gamma \geq 0,\ \gamma \neq 1 \\ \ln(C) & \text{if } \gamma = 1\,. \end{cases}$$

Each person inelastically supplies $H > 0$ hours of labor. People of the same age are equally productive. Labor earnings at age t equal $Y_t = Y_0\, e^{gt}$ for $0 \leq t < \tau$, where $g > 0$. Earnings growth reflects increased labor efficiency from experience. Let X denote an agent's wealth process. All agents start life with $X_0 = 0$ and identical labor earnings Y_0. This is the sense in which they are ex ante identical.

As in refs. 5 and 6, we assume that people can purchase an actuarially fair "reverse-life-insurance" contract that provides payments at rate of λX_t until death in exchange for agreeing to transfer end-of-life wealth $X_{\tau-}$ to an insurance company.

A random variable $S_t = 0$ if an agent is alive, and $S_t = 1$ otherwise. For $(0 \leq t < \tau)$, wealth evolves as

$$dX_t = [(r+\lambda)X_{t-} + Y_{t-} - C_{t-}]\,dt - X_{t-}\,dS_t\,. \qquad [2]$$

The term in brackets is the saving rate $\dot{X}_t$. The rate of return on savings equals the sum of the actuarially fair payment rate λ and the risk-free rate r. The term that multiplies dS_t is a transfer of the agent's wealth $X_{\tau-}$ just prior to death to the insurance company at death moment τ, i.e., when $dS_t = 1$.

A person can dissave when savings X_t are positive, but cannot borrow against future labor earnings, i.e.,

$$X_t \geq 0, \quad \text{for all} \quad t \geq 0\,. \qquad [3]$$

Each person maximizes utility functional Eq. **1** subject to the law of motion [**2**] and an associated transversality condition.

We complete our model as did ref. 1 by letting a representative firm operate a Cobb–Douglas production technology. A representative firm operates a production function $F(K,L) = \mathcal{A}K^{\alpha}L^{1-\alpha}$, where $\mathcal{A} > 0$, $\alpha \in (0,1)$, K is the aggregate capital stock, and L is aggregate labor demand. Physical capital depreciates at a constant rate $\delta > 0$. The firm rents capital and labor in competitive markets. The firm's optimization problem implies that a competitive equilibrium interest rate r and wage index w satisfy:

$$r = F_K(K,L) - \delta \text{ and } w = F_L(K,L)\,. \qquad [4]$$

Equilibrium. Across people, random deaths are statistically independent. To sustain a constant population, we replenish the economy with new people born at a constant rate λ per unit of time. By a law of large numbers, the insurance company always breaks even by using its receipts to cover its payments to living annuity owners. In equilibrium, capital demand equals capital supply:

$$K = \mathbb{E}(X) \equiv \int_0^{\infty} X\phi_X(X)\,dX\,, \qquad [5]$$

Significance

Forces that shape wealth inequality are intermediated through an individual's nonfinancial earnings growth rate *g* and an equilibrium interest rate *r*. Individuals' earnings growth rate and survival probability interact with their preferences about consumption plans to determine aggregate savings and the interest rate and make wealth more unequally distributed and have a fatter tail than labor earnings, as in US data.

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*Except that we formulate things in continuous, rather than discrete, time to streamline the mathematics, our model is in an applied tradition initiated by refs. 1 and 2, which built on theoretical work of refs. 3 and 4.

where $\phi_X(X)$ is the cross-section stationary probability density of wealth X.

In equilibrium, labor demand equals labor supply: $L = H$. The wage index w equals an average wage rate across all agents so that $w = \mathbb{E}(Y)/H$. Because aggregate labor cost wL equals aggregate labor earnings, a law of large numbers[†] implies

$$wL = wH = \mathbb{E}(Y) \equiv \int_0^\infty Y\phi_Y(Y)dY\,, \qquad [6]$$

where $\phi_Y(Y)$ is the cross-section stationary distribution of labor earnings. Therefore, an agent's labor earnings Y_t exceeds the average level $\mathbb{E}(Y)$ if and only if her wage rate Y_t/H at t exceeds w.

Eqs. **4**, **5**, and **6** imply that the equilibrium interest rate r and wage rate w received by an agent with average labor efficiency satisfy

$$r = \mathcal{A}\alpha\left(\frac{K}{L}\right)^{\alpha-1} - \delta = \frac{\alpha}{1-\alpha}\frac{\mathbb{E}(Y)}{\mathbb{E}(X)} - \delta\,, \qquad [7]$$

$$w = \mathcal{A}(1-\alpha)\left(\frac{K}{L}\right)^{\alpha} = \mathcal{A}(1-\alpha)\left(\frac{\mathbb{E}(X)}{H}\right)^{\alpha}. \qquad [8]$$

We calculate a cross-section marginal stationary distribution of wealth by integrating a cross-section joint distribution of wealth and earnings. Where $C(X, Y)$ is a decision rule for consumption and $\mu_X(X, Y) = (r+\lambda)X + Y - C(X, Y)$, the following Kolmogorov Forward (Fokker–Planck) equation describes the cross-section joint distribution $\phi_{XY}(X, Y)$:

$$\lambda\phi_{XY}(X, Y) = -\frac{\partial(\mu_X(X, Y)\phi_{XY}(X, Y))}{\partial X} - \frac{\partial(gY\phi_{XY}(X, Y))}{\partial Y}\,. \qquad [9]$$

A stationary recursive competitive equilibrium consists of value functions, decision rules for consumption, an interest rate r, an average wage rate w, stationary population demographics, and a stationary distribution for a cross-section joint distribution for wealth and earnings (X, Y) such that

1. Given r and the labor-earnings process $\{Y_s : s \geq 0\}$ and X_0, decision rules solve each person's lifetime savings problem;
2. The interest rate r and wage index w satisfy [**7**] and [**8**];
3. Eqs. **5** and **6** hold so that markets for capital and labor clear;
4. The cross-section distribution of wealth and earnings $\phi_{XY}(X, Y)$ is time-invariant and satisfies Eq. **9**.

Computing Equilibria

We provide analytic formulas to isolate forces that determine equilibrium outcomes.[‡] .

To assure existence of equilibrium objects, we assume

$$\lambda > g \geq 0, \qquad [10]$$

so that the death rate λ exceeds the earnings growth rate g that exceeds zero. To compute a stationary equilibrium, we proceed as follows. We start from an exogenous cross-section distribution of labor earnings governed by a power law. Next, for a given interest rate that is consistent with positive aggregate savings, we use an optimal decision rule for consumption together with budget constraints to deduce dynamics of each person's wealth and an implied cross-section distribution of wealth. Then, we compute an interest rate that equates aggregate supplies of labor and capital to aggregate quantities that firms demand.

Cross-Section Earnings Distribution. Labor earnings grow according to $Y_t = Y_0\,e^{gt}$, and length of life is the only source of heterogeneity across people. Along with Condition [**10**], a constant mortality rate λ implies that the cumulative distribution function (CDF) of the cross-section of earnings is

$$\Phi_Y(Y) = 1 - \left(\frac{Y}{Y_0}\right)^{-\xi_Y}, \qquad [11]$$

where

$$\xi_Y = \lambda/g\,, \qquad [12]$$

with mean

$$\mathbb{E}(Y) = \int_0^\infty Y d\Phi_Y(Y) = \frac{\lambda}{\lambda - g}Y_0\,. \qquad [13]$$

Evidently, $\Phi_Y(Y)$ has a fat tail with a power-law exponent $\xi_Y = \lambda/g > 1$. Researchers including refs. 9–15 also combined exponential growth with a constant exit rate to attain such results.

Eq. **11** implies the following Lorenz curve of labor earnings:

$$\mathcal{L}_Y(z) \equiv \frac{\int_0^z \Phi_Y^{-1}(u)du}{\int_0^1 \Phi_Y^{-1}(u)du} = 1 - (1-z)^{\frac{\lambda-g}{\lambda}}\,. \qquad [14]$$

The fraction of labor earnings earned by the top $(10 \times u)$ percent of people that goes to the top u percent is constant:

$$FI_Y(u) \equiv \frac{1 - \mathcal{L}_Y(1 - 0.01 \times u)}{1 - \mathcal{L}_Y(1 - 0.1 \times u)} = 10^{\frac{g}{\lambda}-1} > 0.1\,. \qquad [15]$$

Condition [**10**] ($\lambda > g$) implies $FI_Y(u) > 0.1$, which means that earnings have a fat right tail with a constant FI for all admissible levels of u.

By using Eq. **14**, we obtain the following formula for the Gini coefficient of labor earnings:

$$\Gamma_Y \equiv 2\int_0^1 (z - \mathcal{L}_Y(z))dz = \frac{g}{2\lambda - g}\,. \qquad [16]$$

Inequalities [**10**] imply $0 \leq \Gamma_Y < 1/2$. The higher the earnings growth g, the larger the earnings inequality. For the special case with no growth ($g = 0$), $\Gamma_Y = 0$.

Optimal Consumption. A scalar

$$q = \frac{1}{r + \lambda - g}, \qquad [17]$$

converts a unit of labor-earnings Y_t into human wealth qY_t in the sense of ref. 16. Thus,

$$P_t = X_t + qY_t, \qquad [18]$$

becomes the sum of financial and human wealth. Total wealth $\{P_t; t \geq 0\}$ serves as s single state variable that determines a person's lifetime utility when $X_t > 0$ at all $t > 0$ before death.[§] Because the market structure allows people to hedge mortality risk, the optimal consumption rule is linear in total wealth P_t:

$$C_t = mP_t = m\,(X_t + qY_t), \qquad [19]$$

[†]See ref. 7 for technical conditions under which we can construct the associated probability and agent measures that allow invoking a law of large numbers.

[‡]Ref. 8 uses a related model to link technology to the personal income and wealth distributions in their analysis of the distributional effects of automation.

[§]We shall verify that in equilibrium, $X_t > 0$ at all ages.

where m is the marginal propensity to consume (MPC):

$$m = \rho + \lambda + (1 - \gamma^{-1})(r - \rho). \quad [20]$$

Consumption C_t and total wealth P_t both grow exponentially at a rate $\left(\frac{r-\rho}{\gamma}\right)$ that equals the product of wedge $(r - \rho)$ and the elasticity of intertemporal substitution $1/\gamma$:

$$\frac{dC_t}{C_t} = \frac{dP_t}{P_t} = \left(\frac{r-\rho}{\gamma}\right) dt. \quad [21]$$

Therefore, $P_t = X_t + qY_t = P_0\, e^{(r-\rho)t/\gamma}$ and $C_t = mP_0\, e^{(r-\rho)t/\gamma}$. Below, we call decision rule **[19]** a *Ramsey* rule.

Wealth as a Function of Earnings. A person's age t is tied to her earnings by $t = \frac{\ln(Y_t/Y_0)}{g}$, and wealth X_t at age t satisfies

$$X_t = X(Y_t) = qY_0 \left[\left(\frac{Y_t}{Y_0}\right)^{\frac{r-\rho}{\gamma g}} - \frac{Y_t}{Y_0}\right]. \quad [22]$$

Because X_t is positive at all t, we know that $X(Y)$ is increasing in Y. For $X(Y) > 0$ and $X'(Y) > 0$ on $(0, +\infty)$, it is necessary that $\frac{r-\rho}{\gamma} > g$, i.e.,

$$r > \rho + \gamma g \equiv r_{ramsey}. \quad [23]$$

So an equilibrium interest rate r has to exceed ρ, an agent's discount rate.[¶] That condition is violated by the $r < \rho$ equilibrium outcome in refs. 1, 2, and 4 models with infinitely lived agents. Indeed, **[23]** asserts something even stronger, namely, that the interest rate r must exceed r_{ramsey}, the *augmented golden rule* interest rate for a Ramsey nonstochastic optimal growth model.

Inequality **[23]** also implies that the growth rate of consumption exceeds the growth rate of earnings, a consequence of a constant MPC out of total wealth and the existence of stationary equilibrium. Since inequality **[23]** holds, Eq. **22** implies that wealth is a convex function of earnings. That shape amplifies wealth inequality relative to earnings inequality.

Cross-section wealth is less equally distributed and has a fatter tail than nonfinancial earnings because individuals' optimal saving choices make their financial wealth always grow at a faster rate than nonfinancial earnings. Younger people own less financial wealth, so they choose to make their wealth grow at faster rates than do older people. Growth rates of wealth still exceed growth rates of nonfinancial earnings for very old people. The higher growth rate of total wealth than of nonfinancial earnings combines with compound interest to widen the wealth distribution and fatten its right tail relative to earnings.

Cross-Section Wealth Distribution. The inverse function $X(Y)$ presented in Eq. **22** is increasing in Y under Condition **[23]**. In a stationary equilibrium, those who live longer have higher earnings and more wealth. Indeed, the CDF of wealth, which we denote by $\Phi_X(X)$, satisfies $\Phi_X(X_t) = \Phi_Y(Y_t)$, which implies

$$\Phi_X(X) = 1 - \left(\frac{Y(X)}{Y_0}\right)^{-\frac{\lambda}{g}}. \quad [24]$$

Therefore, the mean of cross-section wealth, X, is

$$\mathbb{E}(X) = \frac{\lambda(r-\rho-\gamma g)}{(\lambda\gamma - (r-\rho))(\lambda - g)} \frac{1}{r+\lambda-g} Y_0. \quad [25]$$

The cross-section distribution of wealth X is asymptotically fat-tailed with a power-law exponent

$$\xi_X = \frac{\gamma\lambda}{r-\rho}. \quad [26]$$

This follows from

$$\lim_{X\to\infty} \frac{1-\Phi_X(X)}{\left(\frac{X}{qY_0}\right)^{-\xi_X}} = \lim_{Y\to\infty} \frac{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{g}}}{\left(\frac{X(Y)}{qY_0}\right)^{-\xi_X}}$$
$$= \lim_{Y\to\infty} \frac{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{g}}}{\left(\frac{Y}{Y_0}\right)^{-\xi_X\cdot\frac{r-\rho}{\gamma g}}} = \lim_{Y\to\infty} \frac{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{g}}}{\left(\frac{Y}{Y_0}\right)^{-\frac{\lambda}{g}}} = 1, \quad [27]$$

where the first equality uses **[24]**, the second equality uses **[22]** and inequality **[23]**, and the third equality follows from **[26]**. Researchers, including refs. 8–11, 13, and 14, have obtained similar results.

Unlike the cross-section earnings distribution that satisfies a power law over the entire support of Y, the cross-section wealth distribution satisfies a power law only in the limit as $X \to \infty$. Thus, the fraction of wealth owned by the top $10 \times u$ percent that goes to the top u percent of people, which we denote by $FI_X(u)$, obeys

$$\lim_{u\to 0} FI_X(u) \equiv \frac{1-\mathcal{L}_X(1-0.01\times u)}{1-\mathcal{L}_X(1-0.1\times u)} = 10^{(1/\xi_X)-1}, \quad [28]$$

where ξ_X is given in Eq. **26**.[‖]

From Micro to Macro. Inequality **[23]** and Eq. **26** together imply that $\xi_X < \lambda/g$: Cross-section wealth has a fatter right tail than earnings since the power-law exponent ξ_X of cross-section wealth is smaller than the exponent λ/g of cross-section earnings.

In *Appendix*, we derive the following formula for the Lorenz curve of wealth:

$$\mathcal{L}_X(z) \equiv \frac{\int_0^z \Phi_X^{-1}(u)\,du}{\int_0^1 \Phi_X^{-1}(u)\,du}$$
$$= \frac{\gamma(\lambda-g)}{r-(\rho+\gamma g)}\left(1-(1-z)^{\frac{(\rho+\gamma\lambda)-r}{\lambda\gamma}}\right)$$
$$- \frac{(\rho+\gamma\lambda)-r}{r-(\rho+\gamma g)}\left(1-(1-z)^{\frac{\lambda-g}{\lambda}}\right), \quad [29]$$

and the following formula for the Gini coefficient of wealth:

$$\Gamma_X = 2\int_0^1 (z-\mathcal{L}_X(z))\,dz = \frac{2\gamma\lambda^2 + g(\rho-r)}{(\rho-r+2\gamma\lambda)(2\lambda-g)}. \quad [30]$$

Our Condition **[10]** asserts that $\lambda > g$ implies that the Gini coefficient for wealth is larger than for earnings: $\Gamma_X > \Gamma_Y$.

Fig. 1 illustrates the mechanism that generates a fatter-tailed distribution for cross-section wealth than for earnings. Fig. 1 *A* and *B* show that earnings Y grow at a constant rate g that is lower than the consumption growth rate $(r-\rho)/\gamma$. This occurs because X grows at a nonlinear rate greater than consumption

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[¶]Because mortality risk is fully hedged via an actuarially fair reverse annuity, it is not the source of $r > \rho$.

[‖]Ref. 8 obtains the same power-law exponent in a model of how automation affects earnings and wealth inequality.

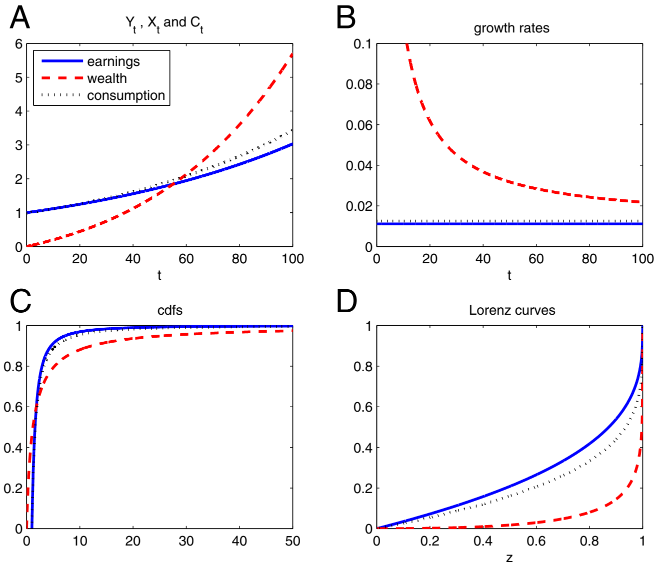

Fig. 1. Earnings, wealth, and consumption: micro dynamics and macro cross-section distribution. *A* plots the levels of Y_t, X_t, and C_t. *B* plots the corresponding growth rate of change over time: $\dot{X}_t/X_t$, $\dot{Y}_t/Y_t$, and $\dot{C}_t/C_t$. *C* plots the CDFs of Y, X, and C. *D* plots the equilibrium stationary cross-section Lorenz curves for Y, X, and C. Parameter values are $\gamma = 2$, $\rho = 5\%$, $\alpha = 0.36$, $\delta = 6\%$, $\mathcal{A} = 0.896$, $\lambda = 0.0167$, $g = 1.1\%$, and $\sigma = 0$.

and earnings growth rates. While the growth rate of wealth $\dot{X}_t/X_t$ decreases with age after starting from ∞ at $X_0 = 0$, $\dot{X}_t$ increases with age.

Fig. 1*C* plots cross-section distributions of Y, X, and C. Fig. 1*D* plots corresponding Lorenz curves. CDFs for both earnings and consumption are described globally by power laws having different exponents. Because our agents prefer to smooth consumption over time, it may at first appear surprising that the distribution of consumption is fatter-tailed than the distribution of earnings. But Condition **[23]** reveals that, in equilibrium, consumption grows faster than earnings. The distribution of wealth is not globally Pareto as earnings and consumption are, but instead approaches the shape of a Pareto distribution with the same power-law exponent as consumption as $X \to \infty$. Fig. 1*D* shows that wealth has a substantially steeper/convex Lorenz curve than consumption, which, in turn, has a steeper/convex Lorenz curve than earnings Y does. Consequently, the Gini coefficient for X is larger than it is for consumption C, which is larger than it is for labor earnings Y.

Note that the Gini coefficient, Lorenz curve, and tail fatness all provide the same inequality rankings for cross-section consumption, labor earnings, and wealth. Such identical rankings won't prevail after we add ex ante heterogeneity across agents.

Aggregate Earnings, Wealth, and Interest Rate. In a stationary equilibrium, $dK_t = d\mathbb{E}(X_t) = 0$. Summing over the wealth dynamics given in Eq. **2** across all agents and using a law of large numbers, we obtain the following relation for aggregate variables:

$$\mathbb{E}(C) = r\mathbb{E}(X) + \mathbb{E}(Y)\,. \qquad [31]$$

$r\mathbb{E}(X)$ is an annuity payment on aggregate wealth, and $\mathbb{E}(Y)$ is aggregate labor earnings. Note that the life-insurance company's transfers to the living equal its receipts from the dying and, hence, do not appear in Eq. **31**.

We have National Income and Product Accounts typical of models in the ref. 1–4 tradition:

$$\begin{aligned} F(K,L) &= F_K(K,L)K + F_L(K,L)L = (r+\delta)K + wL \\ &= (r+\delta)\mathbb{E}(X) + \mathbb{E}(Y) = \mathbb{E}(C) + \delta K\,. \end{aligned} \qquad [32]$$

The first equality follows from Euler's theorem applied to a Cobb–Douglas aggregate production function; the second equality uses the firm's first-order conditions for factors of production: $F_K(K,L) = (r+\delta)$ and $F_L(K,L) = w$; the third equality follows from market-clearing conditions $K = \mathbb{E}(X_t)$ and $wL = wH = \mathbb{E}(Y_t)$; and the fourth equality follows from **[31]**.

To compute a stationary equilibrium interest rate, take the firm's first-order conditions for capital and labor, $r + \delta = F_K(K,L)$ and $w = F_L(K,L)$, and then substitute **[13]** and **[25]** for $\mathbb{E}(Y)$ and $\mathbb{E}(X)$, respectively, to obtain

$$\begin{aligned} r &= \frac{\alpha}{1-\alpha}\frac{wL}{K} - \delta \\ &= \frac{\alpha}{1-\alpha}\frac{\mathbb{E}(Y)}{\mathbb{E}(X)} - \delta = \frac{\alpha}{1-\alpha}\frac{(\rho+\lambda\gamma-r)(r+\lambda-g)}{r-(\rho+\gamma g)} - \delta\,. \end{aligned} \qquad [33]$$

A version of the preceding equation appears also as Eq. **5** and in appendix B.1.1 of ref. 8.

This string of equalities implies a quadratic equation that restricts the stationary equilibrium r:

$$\begin{aligned} \Psi(r) &\equiv r^2 - [\rho + (1-\alpha)(\gamma g - \delta) + \alpha(\gamma\lambda - (\lambda - g))]r \\ &\quad - [(1-\alpha)\delta(\rho+\gamma g) + \alpha(\lambda-g)(\rho+\gamma\lambda)] = 0\,. \end{aligned} \qquad [34]$$

The equilibrium interest rate is the positive root of Eq. **34**.** Evidently,

$$\Psi(0) = -((1-\alpha)\delta(\rho+\gamma g) + \alpha(\lambda-g)(\rho+\gamma\lambda)) < 0\,, \qquad [35]$$

$$\Psi(\rho) = -(1-\alpha)(\rho+\delta)\gamma g - \alpha\gamma\lambda(\rho+\lambda-g) < 0\,, \qquad [36]$$

$$\Psi(\rho+\gamma g) = -(\lambda-g)\alpha\gamma(\rho+\gamma g+\lambda-g) < 0\,, \qquad [37]$$

$$\Psi(\rho+\gamma\lambda) = (\lambda-g)(1-\alpha)\gamma(\rho+\gamma\lambda+\delta) > 0\,. \qquad [38]$$

Fig. 2 reveals that the equilibrium interest rate satisfies:

$$\rho \le \rho + \gamma g < r < \rho + \gamma\lambda\,. \qquad [39]$$

Quantitative Inputs and Outputs

Parameter Choices. To isolate sources of new findings about the equilibrium wealth distribution that our model brings, we purposefully choose consensus parameter values from the literature. Thus, we set commonly used values $\gamma = 2$ and an annual discount rate $\rho = 5\%$. We set preference and production function parameters to values used by refs. 17–19. Following refs. 20 and 21, we set the capital share of national income, α, to 0.36. We set an annual depreciation rate of capital, δ, to 6% to match an estimate of the US depreciation-output ratio reported by ref. 22. We want an aggregate capital-output ratio equal to three, as in refs. 18 and 23, which, in light of Eq. **7**, leads to an equilibrium interest rate r of 6% per annum, as in refs. 17 and 18. Along with refs. 8 and 18 and others, we interpret the equilibrium risk-free rate in our model as a broad measure of average returns on capital. This is why we calibrate an annual (real) risk-free rate to be approximately 6%. We set the productivity parameter $\mathcal{A}$ to 0.9, so that the wage rate w for an agent with the average labor efficiency equals unity (a normalization). We set $\lambda = 0.0167$ in order to set an agent's expected lifetime at $1/\lambda = 60$ years, as in ref. 23.

Eq. **21 implies that a negative r together with $X_0 = 0$ (and, hence, $P_0 = Y_0$) would not cohere with the requirement that $\mathbb{E}(X) = K > 0$.

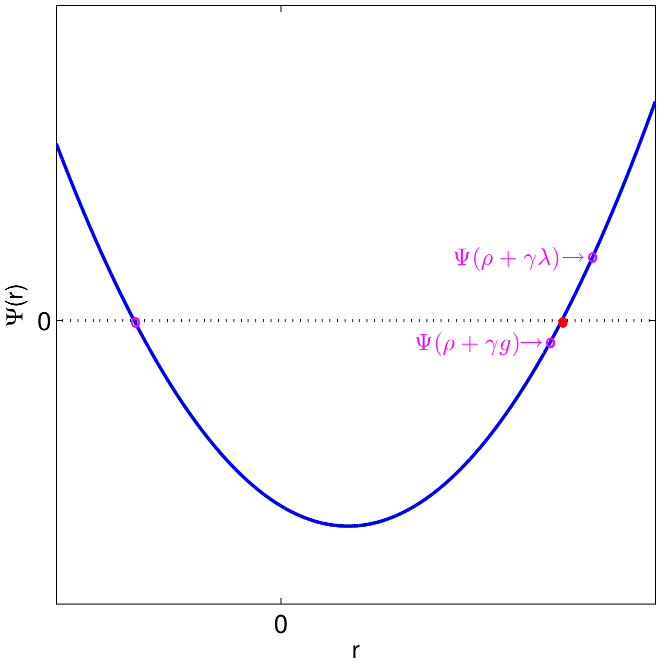

Fig. 2. The quadratic function $\Psi(r)$ given in Eq. **34**. The equilibrium interest rate satisfies $\Psi(r^*) = 0$ and $(\rho + \gamma g) < r^* < (\rho + \gamma\lambda)$ when $\lambda > g \geq 0$, as assumed in Condition [**10**].

Role of Earnings Growth *g*. In Table 1, we conduct a comparative static exercise with respect to the earnings growth rate g. In addition to the equilibrium interest rate r, we report Gini coefficients for earnings and wealth (Γ_Y and Γ_X), power-law exponents (ξ_Y and ξ_X) for the tail, and fractal inequalities (FI_Y and FI_X) for the tail.

First, consider a case with $g = 0$. There is zero cross-section earnings inequality (hence, $\Gamma_Y = 0, \xi_Y = \infty$, and $FI_Y = 0.1$). An equilibrium interest rate $r = 5.88\%$ that exceeds the annual rate of time preference $\rho = 5\%$ makes young people want to save. Our analytical formulas indicate that wealth has a power-law exponent of $\xi_X = \gamma\lambda/(r - \rho) = 2 \times (1/60)/(0.0588 - 0.05) = 3.79$, a Gini coefficient of $\Gamma_X = 1/(2 - \xi_X^{-1}) = 0.58$, and fractal inequality $FI_X = 10^{(1/\xi_X)-1} = 0.18$ for all z. The fraction of wealth earned by the top 0.1% of people that goes to the top 0.01% is 18%, which, because it is larger than 10%, indicates that wealth is fat-tailed.

The $g = 0$ (first) row in Table 1 shows that, in equilibrium, the pure life-cycle savings motive for young people, all of whom are born with no (or small) wealth, can generate a fat-tailed wealth distribution, even when their labor earnings are perfectly equal and wealth inequality is entirely driven by how long different agents live.

At a given interest rate r, a higher labor-earnings growth rate g strengthens incentives to borrow against future income to finance current consumption. To encourage savings and clear the asset market, the equilibrium r must increase with g. Also, as g increases, (aggregate) labor becomes more productive, which raises firms' demand for capital because capital and labor are complements.

Thus, as g increases from 0 to 1%, cross-section earnings inequality increases because older people have higher earnings: The Gini coefficient for cross-section earnings increases from zero to $\Gamma_Y = 1\%/(2 \times (1/60) - 1\%) = 0.43$, and the earnings tail becomes fatter (with the power-law exponent ξ_Y decreasing from ∞ to $\lambda/g = (1/60)/1\% = 1.67$). As a result, the fraction of earnings received by the top 0.01% of agents that goes to the top 0.1% equals 40%: $FI_Y = 10^{1/\xi_Y - 1} = 10^{(1/1.67)-1} = 40\%$, a fraction whose excess over 10% indicates substantial earnings inequality among the earnings-rich.

In order to elicit saving, the equilibrium interest rate increases from 5.88% to 7.34%. When $g = 1\%$, the return on savings is greater than when $g = 0$ because the equilibrium interest rate is higher. As a result, cross-section wealth inequality increases substantially. The Gini coefficient for wealth Γ_X increases to 0.87 from 0.58; the wealth tail becomes fatter with the power-law exponent $\xi_X = \gamma\lambda/(r - \rho)$ decreasing to $2 \times (1/60)/(7.5\% - 5\%) = 1.43$ from 3.79; and the fraction of wealth owned by the top 0.01% of agents owned by the top 0.1% increases to $FI_X = 10^{(1/\xi_X)-1} = 10^{(1/1.43)-1} = 50\%$ from 18%.

Finally, if we adjust parameters to make the Gini coefficient of earnings equal 0.63, the value reported in refs. 18 and 23, we obtain $g = 1.29\%$. In this case, the annual equilibrium interest rate is 7.77%, and the wealth Gini coefficient is 0.95, significantly higher than its value of 0.78 in the US data. What makes cross-section wealth that much fatter-tailed than earnings is that the equilibrium interest rate is high and so many agents live so long, or, if we reinterpret the mortality parameter as partly measuring intergenerational bequest motive, that they care so much about their descendants. Ref. 8 analyzes a setup like this. See their p. 7 discussion on "finite lives and stochastic altruism" and their online appendix B.1.3. Enriching the mortality specification would allow us to improve fits here.[††] .

Table 1 confirms two insights about sources of wealth inequality. First, a higher growth rate of earnings increases Gini coefficients and fattens right tails of both earnings and wealth. Second, for all levels of g, wealth inequality is larger than earnings inequality, whether we measure them with Gini coefficients ($\Gamma_X > \Gamma_Y$), power-law exponents for right tails ($\xi_X < \xi_Y$), or fractal inequalities ($FI_X > FI_Y$). This occurs because older people are both earnings-rich and wealth-rich; their voluntary savings makes their wealth grow at a faster rate than their earnings.[‡‡] However, the result that wealth has a fatter tail than earnings may not hold when there is ex ante heterogeneity.

Table 1 confirms that the equilibrium interest rate r exceeds the earnings growth rate g.[§§] The mechanism here is related to, but distinct from, one posited by ref. 27, which sees an $r > g$ condition as the fulcrum that creates wealth inequality. Unlike ref. 27, our model with its ex post heterogeneous agents explicitly incorporates equilibrium consumption responses of the type analyzed in refs. (1–4). Despite the action of the impatience parameter $\rho > 0$ in making them prefer to front-load their consumption profiles, agents accept upward-sloping consumption profiles that fit together with an equilibrium age-dependent wealth growth rate that is larger than $(r - \rho)/\gamma$, which, in turn, exceeds the earnings growth rate g. These outcomes prevail because, in equilibrium, an individual's wealth accumulates at a rate higher than earnings.

Ex Ante Heterogeneity

Ref. 28 documents that ex ante heterogeneity influences equilibrium wealth distributions. Ref. 29 shows how positing different discount rates across agents can help match equilibrium wealth distributions. We can extend our baseline model to allow for

[††] Refs. 24 and 25 construct a stochastic dynastic/life-cycle model that fits observed demographics reasonably well.

[‡‡] Ref. 26 documents that capital income is more unequally distributed than labor earnings, that the exponent for capital is between one and three, and that it is smaller than the exponent for labor, which is between two and five.

[§§] Along with refs. 8 and 18 and others, we interpret the equilibrium risk-free rate in our model as a broad measure of average returns on capital. This is why we calibrate an annual (real) risk-free rate to be approximately 6%.

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Table 1. Effects of earnings growth rate g

g	r, %	Γ_Y	Γ_X	ξ_Y	ξ_X	FI_Y	FI_X
0	5.88	0	0.58	∞	3.79	0.10	0.18
0.5%	6.60	0.18	0.72	3.34	2.08	0.20	0.30
1%	7.34	0.43	0.87	1.67	1.43	0.40	0.50
1.29%	7.77	0.63	0.95	1.29	1.21	0.59	0.68

Γ_Y and Γ_X are the Gini coefficients for cross-section earnings and wealth, respectively. For all levels of Y, the power-law exponent for earnings is $\xi_Y = \lambda/g$, and the power-law exponent for wealth approaches $\xi_X = \gamma\lambda/(r-\rho)$ as $X \to \infty$.

this and other varieties of ex ante heterogeneity. Thus, suppose that groups of people, A and B, differ in earnings growth rates (g^A and g^B) , elasticity of intertemporal substitution ($1/\gamma^A$ and $1/\gamma^B$), subjective discount rates (ρ^A and ρ^B), or death (or dynasty exit rates) λ^A and λ^B. Let θ denote the population of type-A agents and $(1-\theta)$ denote the population of type-B agents. Assume $\lambda^A > g^A \geq 0$ and $\lambda^B > g^B \geq 0$, so that stationary earnings distributions exist for both groups.

Cross-Section Earnings Distribution. The CDF for cross-section earnings is

$$\Phi_Y(Y) = 1 - \theta\left(\frac{Y}{Y_0}\right)^{-\lambda^A/g^A} - (1-\theta)\left(\frac{Y}{Y_0}\right)^{-\lambda^B/g^B}, \quad [40]$$

and so has a fat tail with a power-law exponent equal to $\min\{\lambda^A/g^A, \lambda^B/g^B\}$. A higher earnings growth rate g or a lower death/exit rate λ makes the tail of the distribution fatter. Using [40], we obtain economy-wide average earnings:

$$\mathbb{E}(Y) = \int_0^\infty Y d\Phi_Y(Y) = \left(\frac{\lambda^A\theta}{\lambda^A - g^A} + \frac{\lambda^B(1-\theta)}{\lambda^B - g^B}\right) Y_0 . \quad [41]$$

Cross-Section Wealth Distribution. Let

$$\pi^N = \frac{g^N\gamma^N}{r-\rho^N}, \quad [42]$$

where $N = A, B$ and

$$\pi^A \geq \pi^B . \quad [43]$$

A financially unconstrained type-N agent optimally chooses a linear Ramsey consumption rule at all t, and her optimal consumption growth rate equals $(r-\rho^N)/\gamma^N$. Fraction π^N is defined as the earnings growth rate g^N divided by this optimal consumption growth rate $(r-\rho^N)/\gamma^N$. But if the no-borrowing constraint binds $X_t \geq 0$, that Ramsey rule cannot be used for all t. Nevertheless, the definition of π^N helps us to characterize the cross-section wealth distribution. We provide conditions under which Ramsey linear consumption rules are optimal for both groups or only for group B and then characterize associated cross-section wealth distributions.

Ramsey rules for both groups. Consider the case where

$$\textbf{Case1}: 1 > \pi^A \geq \pi^B . \quad [44]$$

This condition means that for both groups, Ramsey consumption rules are feasible and optimal. So each agent saves and sees its wealth X_t increasing with age t. Therefore, the earnings growth rate is lower than the consumption growth rate in equilibrium.

The CDF or cross-section wealth for type $N = A, B$ is

$$\Phi^N(X) = 1 - \left(\frac{Y^N(X)}{Y_0}\right)^{-\frac{\lambda^N}{g^N}}, \quad [45]$$

where $Y^N(X)$ is the value of Y that solves $X = X^N(Y)$:

$$X^N(Y) = qY_0\left[\left(\frac{Y}{Y_0}\right)^{1/\pi^N} - \frac{Y}{Y_0}\right]. \quad [46]$$

When $\pi^N < 1$, the power-law exponent for the right tail of wealth distribution (for type-N agents) equals the product of π^N and $\frac{\lambda^N}{g^N}$, which we denote by ξ_X^N:

$$\xi_X^N = \frac{\gamma^N\lambda^N}{r-\rho^N}. \quad [47]$$

As noted in [10], since $\lambda^N > g^N$ is required to ensure that a stationary earnings distribution exists, $\xi_X^N > \pi^N$ is satisfied when $\pi^N < 1$ because $\xi_X^N > 1$.

The CDF of the cross-section wealth distribution is

$$\begin{aligned}\Phi_X(X) &= 1 - \theta\left(1-\Phi^A(X)\right) - (1-\theta)\left(1-\Phi^B(X)\right) \quad [48]\\ &= 1 - \theta\left(\frac{Y^A(X)}{Y_0}\right)^{-\frac{\lambda^A}{g^A}} - (1-\theta)\left(\frac{Y^B(X)}{Y_0}\right)^{-\frac{\lambda^B}{g^B}},\end{aligned}$$

so cross-section wealth is asymptotically fat-tailed with power-law exponent ξ_X

$$\xi_X = \min\left\{\xi_X^A, \xi_X^B\right\}, \quad [49]$$

where ξ_X^N is the power-law exponent of the wealth distribution for $N = A, B$ given in [47]. The wealth-rich who are at the right tail are long-lived ones from the group and have a lower value of the power-law exponent ξ_X^N. That is, people who are more patient (a lower ρ), more willing to substitution consumption over time (a higher elasticity of intertemporal substitution, $1/\gamma$), and/or less likely to die (a lower λ) accumulate more wealth and move into the right tail of the wealth distribution.

Average wealth is

$$\begin{aligned}\mathbb{E}(X) = &\frac{\left(r-\rho^A-\gamma^A g^A\right)}{(\lambda^A\gamma^A-(r-\rho^A))(\lambda^A-g^A)}\frac{\theta\lambda^A}{r+\lambda^A-g^A}Y_0\\ &+\frac{\left(r-\rho^B-\gamma^B g^B\right)}{(\lambda^B\gamma^B-(r-\rho^B))(\lambda^B-g^B)}\frac{(1-\theta)\lambda^B}{r+\lambda^B-g^B}Y_0 .\end{aligned} \quad [50]$$

The equilibrium interest rate, r, solves

$$r = \frac{\alpha}{1-\alpha}\frac{\mathbb{E}(Y)}{\mathbb{E}(X)} - \delta, \quad [51]$$

where $\mathbb{E}(Y)$ is given by [41], and $\mathbb{E}(X)$ is given by [50]. The equilibrium interest rate satisfies

$$\max_{N=A,B}\left(\rho^N + \gamma^N g^N\right) < r < \min_{N=A,B}\left(\rho^N + \gamma^N\lambda^N\right). \quad [52]$$

The left inequality in [52] states that the equilibrium interest rate r exceeds the larger $\left(\rho^N + \gamma^N g^N\right)$ to ensure that savings motives are sufficiently strong for both groups so that they want to use Ramsey rules. An implication of this result is that the equilibrium interest rate $r > \max\{\rho^A, \rho^B\}$, in contrast to outcomes in refs. 1, 2, and 4 models with infinitely lived agents. The right inequality in [52] states that the equilibrium interest rate cannot be so high that it causes consumption for either group to grow at a rate $(r - \rho^N)/\gamma^N$ larger than the death/exit rate λ^N. Otherwise, there is no stationary wealth distribution.

Table 2. Effects of ρ^A: $\pi^A < 1$

ρ^A	r, %	Γ_Y	Γ_X	ξ_Y	ξ_X	FI_Y	FI_X
5%	**7.50**	**0.50**	**0.90**	**1.51**	**1.34**	**0.46**	**0.56**
5.2%	7.59	0.50	0.91	1.51	1.29	0.46	0.60
5.4%	7.65	0.50	0.97	1.51	1.26	0.46	0.62

Γ_Y and Γ_X are the Gini coefficient, ξ_Y and ξ_X are the power-law exponents of the right tail, and FI_Y and FI_X are the fractal inequality of the right tail for cross-section earnings and wealth, respectively. We set $\theta = 0.5$, $g^A = g^B = 1.11\%$, $\lambda^A = \lambda^B = 0.0167$, $\rho^B = 0.05$, $\gamma^A = \gamma^B = 2$, $\delta = 0.06$, and $\alpha = 0.36$.

Group A as hand-to-mouth consumers. Next, we turn to a situation in which one group of agents is financially constrained:

$$\textbf{Case2}: \pi^A \geq 1 > \pi^B. \quad \textbf{[53]}$$

Now, agents in group A with their high earnings growth g, high discount rate ρ, or low elasticity $1/\gamma$ want to front-load consumption enough to cause $X_t \geq 0$ to bind, i.e., $X_t = 0$ at all t. That makes them into hand-to-mouth consumers forever.[¶¶] In equilibrium, agents in the other group cannot be hand-to-mouth consumers, and we must have $\pi^B < 1$. Otherwise, there would be insufficient savings to support aggregate production, driving the marginal product of capital to plus infinity. The equilibrium interest rate has to be at a level where $\pi^B < 1$, meaning that agents in group B are financially unconstrained with optimal consumption functions that take the form of Ramsey rules.

Recall that $X = 0$ for all agents in group A and that their mass is θ. People in group B all have positive savings. Therefore, the CDF for the wealth distribution has positive probability mass at $X = 0$: $\Phi_X(0) = \theta$ and

$$\Phi_X(X) = 1 - (1-\theta)\left(\frac{Y^B(X)}{Y_0}\right)^{-\frac{\lambda^B}{g^B}} \quad \text{for } X > 0. \quad \textbf{[54]}$$

In **[54]**, $Y^B(X)$ is the value of Y that solves $X = X^B(Y)$, where $X^B(Y)$ is given by **[46]** with parameter values for $N = B$. Eqs. **54** and **46** imply that the power-law exponent of wealth is ξ_X^B, where ξ_X^B is given by **[47]** with $N = B$.

Equilibrium average wealth is

$$\mathbb{E}(X) = \frac{\left(r - \rho^B - \gamma^B g^B\right)}{(\lambda^B\gamma^B - (r-\rho^B))(\lambda^B - g^B)} \frac{(1-\theta)\lambda^B}{r + \lambda^B - g^B} Y_0. \quad \textbf{[55]}$$

The equilibrium interest rate, r, solves **[51]**, where $\mathbb{E}(X)$ is given in **[55]** and $\mathbb{E}(Y)$ is given in **[41]**.

The equilibrium interest rate satisfies the following restriction:

$$\rho^B \leq \rho^B + \gamma^B g^B < r < \rho^B + \gamma^B \lambda^B. \quad \textbf{[56]}$$

As group A agents are hand-to-mouth, only parameter values for group B appear in **[56]**. As in our baseline model with no ex ante heterogeneity, the equilibrium interest rate r exceeds the subjective discount rate for the financially unconstrained type-B agents (ρ^B) by $\gamma^B g^B$, but the equilibrium consumption growth rate $(r - \rho^B)/\gamma^B$ cannot exceed death rate λ^B, as required by the inequality on the right side of **[56]**. Otherwise, there is no stationary wealth distribution.

Consequences of Heterogeneous Discount Rates and Earnings Growth Rates. In Tables 2 and 3, we display consequences of varying ρ^A. We hold all other parameters at the same values for the two groups: $\lambda^A = \lambda^B = 0.0167$, $g^A = g^B = 1.11\%$, and $\gamma^A = \gamma^B = 2$. When the subjective discount rate ρ^A is just slightly larger than $\rho^B = 5\%$, which means $\pi^A < 1$ holds (case 1), the equilibrium consumption rules for both groups are linear in wealth and earnings. We report these results in Table 2. The first row corresponds to the baseline case with no heterogeneity as $\rho^A = \rho^B = 5\%$. Therefore, $r = 7.5\%$, and we reproduce results from Table 1. By increasing ρ^A (e.g., to 5.4%) and keeping $\rho^B = 5\%$, type-A agents increase their consumption, and firms would want more capital if r were fixed at 7.5%, so the equilibrium interest rate has to increase (to 7.65%). As a result, a higher interest rate helps savers accumulate wealth, and, hence, wealth inequality widens, as indicated by a higher Gini coefficient Γ_X, a lower power-law exponent ξ_X^N, and a higher fractional inequality FI_X.

As we continue to increase ρ^A to 5.45% (the first row in Table 3), consumption for group A continues to increase up to the point where $\pi^A = 1$, which implies $X_t^A = 0$ for all agents in group A at $t \geq 0$. As a result, for markets to clear, the interest rate again continues to increase. Using **[42]**, we confirm this intuition: the equilibrium interest rate: $r = \rho^A + g^A\gamma^A = 5.45\% + 1.11\% \times 2 = 7.67\%$. Because the interest rate increased only 2 basis points as we increase ρ^A from 5.4% to 5.45%, wealth inequality increases only slightly. Finally, further increasing ρ^A does not change equilibrium outcomes because type-A agents are constrained. This is why the two rows in Table 3 are the same.

Outcomes displayed in Tables 2 and 3 corroborate results of ref. 29, which uses heterogeneous discount rates to generate an empirically plausible wealth distribution.

Tables 4 and 5 reports equilibrium consequences of varying g^A. We keep other parameters identical for the two groups: $\lambda^A = \lambda^B = 0.0167$, $\rho^A = \rho^B = 0.05$, and $\gamma^A = \gamma^B = 2$. When the earnings growth rate g^A is not too high so that $\pi^A < 1$ holds (case 1), equilibrium consumption rules for agents in both groups are linear in wealth and earnings. We report these results in Table 4. The first row corresponds to the baseline case with no heterogeneity as $g^A = g^B = 1.11\%$ and, thus, $r = 7.5\%$, and we recover results reported in Table 1. By increasing g^A (e.g., to 1.3%) and keeping $g^B = 1.11\%$, type-A agents increase their consumption, and firms would demand more capital (if r were fixed at 7.5%), so the equilibrium interest rate has to increase (to 7.67%). As a result, savers accumulate wealth at a higher interest rate, and wealth inequality widens, as witnessed by a higher Gini coefficient Γ_X, a lower power-law exponent ξ_X^N, and a higher fractional inequality FI_X

As we increase g^A to 1.39% (the first row in Table 5), consumption for group A increases up to where $\pi^A = 1$, which implies $X_t^A = 0$ for all agents in group A at $t \geq 0$. As a result, for markets to clear, the interest rate again has to increase. Using **[42]**, we confirm this reasoning: the equilibrium interest rate: $r = \rho^A + g^A\gamma^A = 5\% + 1.39\% \times 2 = 7.78\%$. Earnings inequality and wealth inequality both also increase.

As we further increase g^A from 1.39 to 1.6%, consumption for agents in group A no longer responds, as they are involuntarily constrained to be hand-to-mouth consumers for any g^A satisfying

Table 3. Effects of ρ^A: $\pi^A \geq 1$

ρ^A	r, %	Γ_Y	Γ_X	ξ_Y	ξ_X	FI_Y	FI_X
5.45%	7.67	0.50	0.97	1.51	1.25	0.46	0.63
6%	7.67	0.50	0.97	1.51	1.25	0.46	0.63

Γ_Y and Γ_X are the Gini coefficient, ξ_Y and ξ_X are the power-law exponents of the right tail, and FI_Y and FI_X are the fractal inequality of the right tail for cross-section earnings and wealth, respectively. We set $\theta = 0.5$, $g^A = g^B = 1.11\%$, $\lambda^A = \lambda^B = 0.0167$, $\rho^B = 0.05$, $\gamma^A = \gamma^B = 2$, $\delta = 0.06$, and $\alpha = 0.36$.

[¶¶]Refs. 30 and 31 refer to consumers who equal consumption to earnings with no savings as hand-to-mouth consumers and document that they constitute a sizable proportion of consumers.

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$1.39\% \leq g^A \leq \lambda^A$. As labor becomes more productive (higher g^A), the firm's demand for capital continues to increase (as capital and labor are complements). In equilibrium, the interest rate rises to 8.06% to restore equilibrium for the case where $g^A = 1.6\%$.

A faster earnings growth rate increases both earnings inequality and the equilibrium interest rate r. Therefore, wealth inequality increases because savers accumulate wealth at a faster rate via a higher r. Thus, faster earnings growth generates larger earnings inequality and also larger wealth inequality. These outcomes are consistent with our baseline analysis with ex ante identical agents.

While inequality measures for earnings and wealth both increase with g^A, whether wealth inequality is greater than earnings inequality for a given g^A depends on how we measure inequality. When $\pi^A > 1$ in equilibrium, the Gini coefficient (i.e., two times the area between the 45° line and the Lorenz curve) and measures of tail fatness (e.g., the power-law exponent and the fractal inequality, FI) yield opposite answers.

Table 5 shows that the earnings distribution has a fatter right tail than does the wealth distribution when $\pi^A > 1$ in equilibrium. For example, when $g^A = 1.6\%$, the power-law exponent for earnings is 1.04, which is lower than the power-law exponent for wealth, 1.09. The fraction of wealth owned by the top $10 \times u$ percent owned by the top u percent of people as u goes to zero, $\lim_{u \to 0} FI_X(u)$, approaches 82%, which is already very large. However, this fractal inequality measure for earnings yields an even worse earnings inequality, $FI_Y = 91\%$, meaning that the fraction of earnings earned by the top $10 \times u$ percent that goes to the top u percent of people as u goes to zero, $\lim_{u \to 0} FI_X(u) = 0.91$, which is larger than that measure for the wealth distribution.

Nevertheless, the Gini coefficient for earnings is much smaller than the Gini coefficient for wealth: $\Gamma_Y = 0.68$ versus $\Gamma_X = 0.98$ for the case where $g^A = 1.6\%$. This is because group A agents are at the bottom of the wealth distribution, with zero wealth, which substantially increases the Gini coefficient for wealth, while their being at the left tail evidently has no effect on the right tail of the wealth distribution. Indeed, the wealth-rich are people who have lower earnings growth, but who have lived long.

In Fig. 3, we plot the Lorenz curves for earnings and wealth in A and B, respectively. We see that as we increase g^A, Lorenz curves for both earnings and wealth become steeper, and the Gini coefficient also increases.

Concluding Remarks

Our paper shares topics, but not models, methods, or findings, with ref. 27. Ref. 27 bristles with fascinating claims about sources of wealth inequalities and presents them to a broad audience by deploying what ref. 32 called "implicit theorizing" that can leave a technically inclined reader not knowing assumptions that make things fit together. Parts of his argument that lead him to emphasize an "$r > g$" condition as a cause of cross-section dispersion in wealth rest on an appeal to a single-agent growth model that has no wealth or income inequality.

Our model tightly links outcomes for individuals to macroeconomic outcomes that include the celebrated "r and g" variables that concerned ref. 27. A wedge between an equilibrium *growth rate* for wealth/savings, $g_t^X \equiv \dot{X}_t / X_t$ and an earnings growth rate g at the level of individual people (not at the aggregate level) makes cross-section wealth more unequal than labor earnings.

Mathematics ties together equilibrium model outcomes: The same forces that make cross-section wealth more unevenly distributed and fatter-tailed than cross-section earnings also make an individual's wealth grow at a higher rate than do her earnings. Firms' demand for physical capital and an equilibrium growth rate for an individual's savings that exceeds the growth rate of her labor earnings ($g_t^X \geq (r - \rho)/\gamma > g > 0$) imply that the equilibrium interest rate r exceeds the nonstochastic augmented golden-rule interest rate $r_{ramsey} = \rho + \gamma g$.

Our paper shares tools, explicit theorizing, and some, but not all, goals with ref. 8, but differs in focus and details of the formal economic environments being modeled. They study effects of technical change and automation on an equilibrium wealth distribution and derive power laws like ones that we, too, find. Details about insurance arrangements and whether earnings processes are stationary or display growth differ between their framework and ours. What unites our project and theirs is our common reliance on the same mathematical tools for characterizing outcomes in heterogeneous-agent models cast in continuous time in closed forms.*** .

To obtain an enlightening and interpretable explicit solution for the wealth distribution, we have analyzed an admittedly unrealistic model with no shocks to labor earnings. Uninsurable shocks to labor earnings are, of course, important, so in ref. 25, we incorporate permanent uninsurable Brownian shocks to labor earnings. Quantitative outcomes in that model depend crucially on earnings growth volatility and precautionary savings. That model can be used to study how government tax and transfer policies affect equilibrium outcomes.

Table 4. Effects of g^A: $\pi^A < 1$

g^A	*r*, %	Γ_Y	Γ_X	ξ_Y	ξ_X	FI_Y	FI_X
1.11%	7.50	0.50	0.90	1.51	1.34	0.46	0.56
1.2%	7.57	0.53	0.91	1.39	1.30	0.52	0.59
1.3%	7.67	0.56	0.94	1.28	1.25	0.60	0.63

Γ_Y and Γ_X are the Gini coefficients, ξ_Y and ξ_X are the power-law exponents of the right tail, and FI_Y and FI_X are the fractal inequality of the right tail for cross-section distribution for earnings and wealth, respectively. We set $\theta = 0.5$, $g^B = 1.11\%$, $\lambda^A = \lambda^B = 0.0167$, $\rho^A = \rho^B = 0.05$, $\gamma^A = \gamma^B = 2$, $\delta = 0.06$, and $\alpha = 0.36$.

Table 5. Effects of g^A: $\pi^A \geq 1$

g^A	*r*, %	Γ_Y	Γ_X	ξ_Y	ξ_X	FI_Y	FI_X
1.39%	7.78	0.60	0.97	1.20	1.20	0.68	0.68
1.5%	7.88	0.64	0.97	1.11	1.16	0.79	0.73
1.6%	8.06	0.68	0.98	1.04	1.09	0.91	0.82

Γ_Y and Γ_X are the Gini coefficients, ξ_Y and ξ_X are the power-law exponents of the right tail, and FI_Y and FI_X are the fractal inequality of the right tail for cross-section distribution for earnings and wealth, respectively. We set $\theta = 0.5$, $g^B = 1.11\%$, $\lambda^A = \lambda^B = 0.0167$, $\rho^A = \rho^B = 0.05$, $\gamma^A = \gamma^B = 2$, $\delta = 0.06$, and $\alpha = 0.36$.

Appendix

Cross-Section Earnings Distribution. The Kolmogorov Forward equation for the density function $\phi_Y(Y)$ is

$$0 = -\left(\frac{\partial(gY\phi_Y(Y))}{\partial Y}\right) - \lambda\phi_Y(Y). \qquad \textbf{[57]}$$

The density function implied by [**57**] is

$$\phi_Y(Y) = \frac{\lambda}{gY_0}\left(\frac{Y}{Y_0}\right)^{-\frac{g+\lambda}{g}}, \qquad \textbf{[58]}$$

which implies the CDF given in [**11**].

***Ref. 33 analyzed a continuous-time version of a model like refs. 1–4, except with agents who have negative exponential utility generalized to incorporate a discount function proposed in ref. 34, and a labor-earnings process that is affine like the processes defined in ref. 35. Because agents live forever, cross-sectional wealth is less fat-tailed than cross-sectional earnings. Ref. 36 shows that the permanent-income hypothesis holds in the sense that optimal consumption is a martingale in equilibrium.

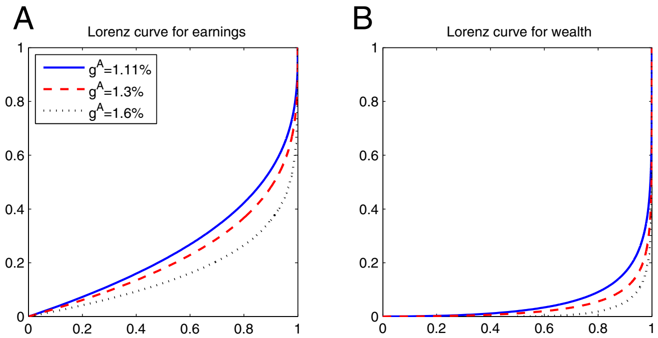

Fig. 3. Lorenz curves for cross-section earnings (*A*) and wealth (*B*). We set $\theta = 0.5$, $g^B = 1.11\%$, $\lambda^A = \lambda^B = 0.0167$, $\rho^A = \rho^B = 0.05$, $\gamma^A = \gamma^B = 2$, $\delta = 0.06$, and $\alpha = 0.36$.

Cross-Section Wealth Distribution. Since the total wealth $P_t = 0$ at the stochastic death moment $t = \tau$, we can rewrite **[21]** as follows for $t < \tau$:

$$dP_t = \left(\frac{r-\rho}{\gamma}\right) P_{t-}\, dt - P_{t-}\, dS_t \, . \qquad \textbf{[59]}$$

Applying the Kolmogorov Forward Equation to $P_t = P(X_t, Y_t)$, we obtain

$$0 = -\frac{\partial}{\partial P}\left[\left(\frac{r-\rho}{\gamma} P\right)\phi_P(P)\right] - \lambda \phi_P(P) \, . \qquad \textbf{[60]}$$

By solving **[60]**, we obtain the following cross-section stationary distribution of P:

$$\phi_P(P) = \xi_X P_0^{\xi_X} P^{-\xi_X - 1} \, , \qquad \textbf{[61]}$$

where $P_0 = qY_0 = Y_0/(r + \lambda - g)$ and ξ_X is given by **[26]**. Eq. **61** implies the following CDF for P:

$$\Phi_P(P) = 1 - \left(\frac{P}{P_0}\right)^{-\xi_X} \, . \qquad \textbf{[62]}$$

Next, compute the inverse of the CDF $\Phi(X)$ for wealth X. Rewriting **[24]** yields

$$\frac{Y(X)}{Y_0} = (1 - \Phi_X(X))^{-\frac{g}{\lambda}} \, . \qquad \textbf{[63]}$$

Substituting **[63]** into **[22]**, we obtain

$$X = \left[\left((1 - \Phi_X(X))^{-\frac{g}{\lambda}}\right)^{\frac{r-\rho}{\gamma g}} - (1 - \Phi_X(X))^{-\frac{g}{\lambda}}\right] qY_0$$
$$= \left[(1 - \Phi_X(X))^{\frac{\rho - r}{\lambda\gamma}} - (1 - \Phi_X(X))^{-\frac{g}{\lambda}}\right] qY_0 \, . \qquad \textbf{[64]}$$

Let $u = \Phi_X(X)$. Rewriting **[64]**, we obtain $X = \Phi_X^{-1}(u)$, where

$$\Phi_X^{-1}(u) = \left((1-u)^{\frac{\rho - r}{\lambda\gamma}} - (1-u)^{-\frac{g}{\lambda}}\right) qY_0 \, . \qquad \textbf{[65]}$$

Integrating $\Phi_X^{-1}(\cdot)$ from zero to z yields

$$\int_0^z \Phi_X^{-1}(u)\, du$$
$$= \left[\frac{\lambda\gamma\left(1 - (1-z)^{\frac{\rho - r + \lambda\gamma}{\lambda\gamma}}\right)}{\rho - r + \lambda\gamma} - \frac{\lambda\left(1 - (1-z)^{\frac{\lambda - g}{\lambda}}\right)}{\lambda - g}\right] qY_0 \, . \qquad \textbf{[66]}$$

We use Eq. **66** when calculating the wealth Lorenz curve and Gini coefficient.

Data Availability. All study data are included in the article.

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